Lecture 21

Coordinates + Linear transformation

$\boxed{\text{e.g}}$ $T: \mathbb{R}^2 \to \mathbb{R}^2 \qquad T(\vec{v}) = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix} \vec{x}$

$\beta = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ = basis for $\mathbb{R}^2$

Suppose $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, i.e. $T(\vec{v}) = A\vec{v}$ for some $n \times n$ matrix A.

- $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$.

$\vec{v} \in \mathbb{R}^n \to \vec{v} = x_1\vec{v}_1 + \ldots + x_n\vec{v}_n$ so set $[\vec{v}]_\beta = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$\vec{v} = S[\vec{v}]_\beta$ where $S_\beta = [\vec{v}_1 \ldots \vec{v}_n]$ ($n \times n$ matrix)

Commutative Diagram

$$\begin{tikzcd} \vec{v} \in \mathbb{R}^n \arrow[r, "A"] & T(\vec{v}) \in \mathbb{R}^n \\ {[\vec{v}]_\beta} \in \mathbb{R}^n \arrow[u, "S"] \arrow[r, "B = S^{-1}AS"'] & {[T(\vec{v})]_\beta} \in \mathbb{R}^n \arrow[u, "S"'] \end{tikzcd}$$

Theorem: Suppose $T: \mathbb{R}^n \to \mathbb{R}^n$ is linear with matrix A and $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$. Then $\forall \vec{v} \in \mathbb{R}^n$ the β-coords of $\vec{v}$ and $T(\vec{v})$ are related by $[T(\vec{v})]_\beta = B[\vec{v}]_\beta$ where $B =$ $n \times n$ matrix given by

$$B = S_\beta^{-1} A S_\beta = \begin{bmatrix} [T(v_1)]_\beta & \cdots & [T(v_n)]_\beta \end{bmatrix}$$

- We call B the <u>B-matrix</u> for T (and A the <u>standard matrix</u> for T).

$\boxed{\text{e.g.}}$

$T: \mathbb{R}^2 \to \mathbb{R}^2 \quad T(\vec{v}) = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix} \vec{x}$

$\beta = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ = basis for $\mathbb{R}^2$

β-matrix for T is:

$$\beta = S_\beta^{-1} A S_\beta = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

e.g. From last time

$P = \text{span}\{\vec{v}_1, \vec{v}_2\} \subset \mathbb{R}^3$

Also $v_3 \in \mathbb{R}^3$ is $\perp$ to $\vec{v}_1, \vec{v}_2$

$\rightarrow \beta = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ = basis for $\mathbb{R}^3$

$T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T(\vec{v})$ = orthogonal projection of $\vec{v}$ onto plane.

β-matrix for T is

$$B = \begin{bmatrix} [T(\vec{v}_1)]_\beta & [T(\vec{v}_2)]_\beta & [T(\vec{v}_3)]_\beta \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} [\vec{v}_1]_\beta & [\vec{v}_2]_\beta & [\vec{0}]_\beta \end{bmatrix}}_{\text{only because we are dealing with orthogonal projection.}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Also $S_\beta B S_\beta^{-1} = A$

↳ note A is standard matrix

↳ note B is the <u>B-matrix</u>

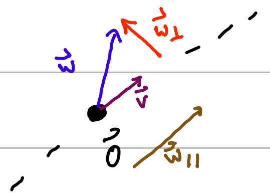
Def - We say that $n \times n$ matrices A + B are similar if $\exists$ invertible matrix S such that

$B = S^{-1}AS$ (i.e: $A = SBS^{-1}$)

If A + B are similar then all powers of A and B are similar.

i.e. $A^k = SB^kS^{-1}$

5.1 Orthogonality (revisited)



$$\vec{w}_{\parallel} = \frac{\vec{w}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\ \vec{v}$$

$$\|\vec{w}_{\parallel}\| = \|\vec{v}\| \frac{\|\vec{w}\cdot\vec{v}\|}{\|\vec{v}\cdot\vec{v}\|}$$

$$= \frac{|\vec{w}\cdot\vec{v}|}{\|\vec{v}\|}$$

Def: The angle between non-zero vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ is the number $\theta \in [0, \pi]$ satisfying:

$$\cos\theta = \frac{\vec{w}\cdot\vec{v}}{\|\vec{w}\|\|\vec{v}\|}$$

$\boxed{\text{e.g}}$

$\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}$

$$\cos\theta = \frac{32}{\sqrt{15}\cdot\sqrt{4^2+5^2+6^2}}$$

Rmk: $\forall\ \vec{v}, \vec{w} \in \mathbb{R}^n$ we have $|\vec{v}\cdot\vec{w}| \leq \|\vec{v}\|\ \|\vec{w}\|$

(Cauchy - Schwarz inequality)